

Unbounded Functions and Positive Linear Operators

J. J. SWETITS

*Department of Mathematics,
Old Dominion University,
Norfolk, Virginia 23508*

AND

B. WOOD

*Department of Mathematics,
University of Arizona,
Tucson, Arizona 85721*

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The approximation of unbounded functions by positive linear operators under multiplier enlargement is investigated. It is shown that a very wide class of positive linear operators can be used to approximate functions with arbitrary growth on the real line. Estimates are given in terms of the usual quantities which appear in the Shisha–Mond theorem. Examples are provided.

1. INTRODUCTION

In recent years there has been a great amount of research concerning the approximation of unbounded functions by means of positive linear operators. In particular, we cite works of Hsu [7, 8], Hsu and Wang [9], Walk [19], Müller and Walk [14], Eisenberg and Wood [4, 5], Ditzian [3], and Rathore [15].

In the papers [4, 5, 7, 8, 9], the technique of multiplier enlargement was employed. Let $\{L_n\}$ be a sequence of linear operators mapping $C[-a, a]$ to $C[-b, b]$ such that L_n is positive on $[-b, b]$, or mapping $C[0, a]$ to $C[0, b]$ such that L_n is positive on $[0, b]$, where $0 < b \leq a$. Let $\{\alpha_n\}$ be a sequence of positive numbers which is strictly increasing to infinity. For $f \in C(-\infty, \infty)$ and $x \in (-\infty, \infty)$, or $f \in C[0, \infty]$ and $x \in [0, \infty]$, the sequence $\{L_n\}$ takes the form $L_n(f(\alpha_n t), x/\alpha_n)$ and this modified operator is positive for all n so large that $|x/\alpha_n| \leq b$.

In the papers cited above, convergence results for unbounded functions

were obtained under the assumption that the functions satisfied some growth condition (e.g., exponential type). An exception to this was noted by Hsu [8]. Let $H_n(f; x)$ be the Hermite–Fejér interpolatory polynomial of degree $\leq 2n - 1$. Then

$$H_n(f, x) = \sum_{k=1}^n f(x_{kn})(1 - xx_{kn}) \left(\frac{T_n(x)}{n(x - x_{kn})} \right)^2, \quad (1.1)$$

where $x_{kn} = \cos((2k - 1)/2n)\pi$, $k = 1, 2, \dots, n$, are the zeros of the Tschebycheff polynomial $T_n(x)$. Hsu showed that, for any $f \in C(-\infty, \infty)$.

$$\lim_{n \rightarrow \infty} H_n \left(f(\alpha_n t); \frac{x}{\alpha_n} \right) = f(x)$$

uniformly on compact subsets of $(-\infty, \infty)$, provided that $\{\alpha_n\}$ was properly chosen. An estimate for the rate of convergence was obtained by Eisenberg and Wood [5].

In [8], Hsu pointed out that, for the Bernstein polynomials and the Landau polynomials, convergence properties have been established only for certain classes of functions with restricted orders of growth along the real axis. He then stated that perhaps the Hermite–Fejér polynomials may be the most suitable ones that can be conveniently modified so as to approximate arbitrary non-bounded, continuous functions on $(-\infty, \infty)$. In this paper we show that such is not the case. We prove, in Section 2, that a very broad class of positive linear operators can be modified by multiplier enlargement so as to approximate any function which is continuous on $(-\infty, \infty)$. Examples are given in Section 3.

2. MAIN RESULT

In the sequel let $e_k(x) = x^k$, $k = 0, 1, 2, \dots$. We note the following. As a consequence of the Riesz representation theorem, if $\{L_n\}$ is a sequence of positive linear operator mapping $C[a, b]$ to $C[c, d]$, where $[c, d] \subseteq [a, b]$, then, for each n , $L_n(f(t), x)$ involves values of $f(t)$ only for $t \in [a, b]$. This fact insures that, if f is continuous on $(-\infty, \infty)$, then the restriction of f to $[a, b]$ is continuous and the growth rate of f is not a factor in determining whether or not $L_n(f)$ exists for each n . Specific examples of such operators are the Bernstein polynomials, the Hermite–Fejér polynomials and the variation-diminishing splines of Marsden and Schoenberg [12, 13, 4].

We shall have need of the following lemmas.

LEMMA 2.1 (compare with [5]). *Let $\{\alpha_n\}$ be a positive sequence, strictly*

increasing to ∞ . Let Ω and its derivative, Ω' , be positive, increasing functions on $[0, \infty]$. If $p > 1$ and $g(x) = [\Omega(\alpha_n|x|)]^p$ for $x \in [-a, a]$, then

$$\omega(g, h) \leq h\alpha_n p[\Omega(\alpha_n a)]^{p-1} \Omega'(\alpha_n a).$$

Here $\omega(g, h)$ denotes the modulus of continuity of g on $[-a, a]$ and $0 < h < a$.

Proof. Since $\Omega^p(x)$ is convex, it follows that for $g(x) = \Omega^p(\alpha|x|)$, $|x| \leq a$,

$$\begin{aligned} \omega(g, h) &= \Omega^p(\alpha a) - \Omega^p(\alpha(a-h)) \\ &= ah p \Omega^{p-1}(\zeta) \Omega'(\zeta), \end{aligned}$$

where $\alpha(a-h) \leq \zeta \leq \alpha a$. Since Ω and Ω' are increasing, it follows that

$$\omega(g, h) \leq ah p \Omega^{p-1}(\alpha a) \Omega'(\alpha a).$$

LEMMA 2.2. Let $\{L_n\}$ be a sequence of linear operators that map $C[-a, a]$ to $C[-b, b]$, $0 < b < a$, such that, for each n , L_n is positive on $[-b, b]$. Let Ω and its derivative, Ω' , be positive, increasing functions on $[0, \infty]$, and let $\{\alpha_n\}$ be a positive sequence, strictly increasing to ∞ . Let $-\infty < \alpha \leq x \leq \beta < +\infty$ and choose N so that $x/\alpha_n \in [-b, b]$ for any $n > N$ and for all $x \in [\alpha, \beta]$. If $p > 1$ and if $n > N$, then

$$\begin{aligned} L_n(\Omega^p(\alpha_n|t|), x/\alpha_n) &\leq \Omega^p(|x|)(1 + \|L_n(e_0) - 1\|) \\ &\quad + (\|L_n(e_0)\| + 1) \mu_n \alpha_n p \Omega^{p-1}(\alpha_n a) \Omega'(\alpha_n a), \end{aligned}$$

where $\mu_n^2 = \max\{L_n((t-y)^2, y) : -b \leq y \leq b\}$ and the norms are sup norms over $[-b, b]$.

Proof. Let $g_n(t) = \Omega^p(\alpha_n|t|)$ and let $n > N$. Since $g_n(t) \in C[-a, a]$, $x/\alpha_n \in [-b, b]$, and L_n is positive on $[-b, b]$, we can apply the theorem of Shisha and Mond [16] to obtain

$$\begin{aligned} \left| L_n \left(g_n(t), \frac{x}{\alpha_n} \right) - g_n \left(\frac{x}{\alpha_n} \right) \right| &\leq \left(1 + L_n \left(e_0, \frac{x}{\alpha_n} \right) \right) \omega(g_n, \mu_n) \\ &\quad + \left| g_n \left(\frac{x}{\alpha_n} \right) \right| \cdot \left| 1 - L_n \left(e_0, \frac{x}{\alpha_n} \right) \right|, \end{aligned}$$

where $\omega(g_n, \cdot)$ is the modulus of continuity of g_n over $[-a, a]$. From Lemma 2.1,

$$\omega(g_n, \mu_n) \leq \mu_n \alpha_n p \Omega^{p-1}(\alpha_n a) \Omega'(\alpha_n a).$$

The proof of the lemma is completed by observing that $g_n(x/\alpha_n) = \Omega^p(|x|)$.

We can now establish our main result.

THEOREM 2.3. *Let $f \in C(-\infty, \infty)$ and let Ω , and its derivative, Ω^1 , be positive increasing functions on $[0, \infty)$. Let $p > 1$ and $1/p + 1/p' = 1$. Assume $f(t) = O(\Omega(|t|)|t|^{2/p'} + t^2)$ ($|t| \rightarrow \infty$), and let $\{\alpha_n\}$ be a positive sequence, strictly increasing to ∞ . Let $\{L_n\}$ be a sequence of linear operators from $C[-a, a]$ to $C[-b, b]$, $0 < b < a$, such that L_n is positive on $[-b, b]$. Let $\delta > 0$, $-\infty < \alpha - \delta < \alpha \leq x \leq \beta < \beta + \delta < \infty$, and choose N so large that $x/\alpha_n \in [-b, b]$. Then, for $x \in [\alpha, \beta]$ and $n > N$,*

$$\begin{aligned} & \left| L_n \left(f(\alpha_n t), \frac{x}{\alpha_n} \right) - f(x) \right| \\ & \leq \omega_\delta(f, \alpha_n \mu_n)(\|L_n(e_0)\| + 1) \\ & \quad + (\alpha_n \mu_n)^2 \left(\frac{\|f\|}{\delta^2} + \frac{C_1 M}{\delta^2} \right) + \|f\| \cdot \|L_n(e_0) - 1\| \\ & \quad + \frac{C_2 M^{1/p'}}{\delta^{2/p'}} (\alpha_n \mu_n)^{2/p'} \{ \Omega^p(|x|)(1 + \|L_n(e_0) - 1\|) \\ & \quad + (\|L_n(e_0)\| + 1) \mu_n \alpha_n p \Omega^{p-1}(\alpha_n a) \Omega'(\alpha_n a) \}^{1/p}, \end{aligned}$$

where $\omega_\delta(f, \cdot)$ denotes the modulus of continuity of f over $[\alpha - \delta, \beta - \delta]$, $\|L_n(e_0) - 1\|$ and $\|L_n(e_0)\|$ are uniform norms over $[-b, b]$, $\|f\|$ is the uniform norm over $[\alpha, \beta]$, C_1, C_2 are positive constants which depend only on f and M is a positive constant which depends only on δ, α and β .

Proof. Minor modifications to Theorem 2 of [15, p. 102], yield

$$\begin{aligned} & \left| L_n \left(f(\alpha_n t), \frac{x}{\alpha_n} \right) - f(x) \right| \\ & \leq \omega_\delta(f, \alpha_n \mu_n)(\|L_n(e_0)\| + 1) \\ & \quad + (\alpha_n \mu_n)^2 \left(\frac{\|f\|}{\delta^2} + \frac{C_1 M}{\delta^2} \right) + \|f\| \cdot \|L_n(e_0) - 1\| \\ & \quad + \frac{C_2 M^{1/p'}}{\delta^{2/p'}} (\alpha_n \mu_n)^{2/p'} \left(L_n \left(\Omega^p(\alpha_n |t|), \frac{x}{\alpha_n} \right) \right)^{1/p}. \end{aligned}$$

Applying Lemma 2.2 to $L_n(\Omega^p(\alpha_n |t|), x/\alpha_n)$ yields the result.

Remarks 1. Our estimate does not require that one evaluate or find an upper bound for $L_n(\Omega^p, x)$. Rather, one only needs the usual quantities $\|L_n(e_0)\|, \|L_n(e_0) - 1\|$ and $L_n((t - x)^2, x)$, as in the Shisha-Mond theorem [16]. Compare with [3, 4, 14, 15].

Remark 2. The theorem is easily modified in case $f \in C[0, \infty)$ and $0 < \alpha < \beta < \infty$. See Section 3.

Remarks 3. Let $\exp_1 x = e^x$ and $\exp_k x = \exp(\exp_{k-1} x)$, $k = 2, 3, 4, \dots$, and let $\ln_1 n = \ln n$, $\ln_K n = \ln(\ln_{K-1} n)$, $K = 2, 3, \dots$. If $\Omega(t) = \exp_K t$ and $\alpha_n = \ln_{K+1} n$ for same K , then $\Omega(\alpha n) = \ln n$ and $\Omega'(\alpha n) = (\ln_1 n) (\ln_2 n) \cdots (\ln_K n) \leq (\ln n)^K$. If $\mu_n = O(n^{-r})$ for same $r > 0$, then our estimate has the simpler form

$$A_1 \omega_\delta \left(f, \frac{l_n n}{n^r} \right) + A_2 \left(\frac{l_n n}{n^r} \right)^2 + \|f\| \cdot \|L_n(e_0) - 1\| + A_3 \left(\frac{l_n n}{n^r} \right)^{2/p'}$$

$$\leq A_1 \omega_\delta \left(f, \frac{l_n n}{n^r} \right) + A_4 \left(\frac{l_n n}{n^r} \right)^{2/p'} + \|f\| \cdot \|L_n(e_0) - 1\|,$$

where $p' > 1$ and $A_i, i = 1, 2, 3, 4$, are constants.

Remarks 4. The multiplier enlargement process worsens the degree of approximation somewhat, i.e., μ_n in the Shisha–Mond theorem [16] becomes $\alpha_n \mu_n$, in Theorem 2.3. Note the example in Section 3.

Remarks 5. Suppose $\{K_n\}$ is a sequence of linear operators from $C[c, d]$ to $C[c + \eta, d - \eta]$, $0 < \eta < (d - c)/2$, such that K_n is positive on $[c + \eta, d - \eta]$. Let g be the linear map from $[-1, 1]$ onto $[c + \eta, d - \eta]$. g^{-1} then maps $[c, d]$ onto $[(c - d)/(d - c - 2\eta), (d - c)/(d - c - 2\eta)]$. Thus, if $f \in C[(c - d)/(d - c - 2\eta), (d - c)/(d - c - 2\eta)]$, the sequence, $\{L_n\}$, where, for $y \in [-1, 1]$,

$$L_n(f(t), y) = K_n(f(g^{-1}(t), g(y))),$$

is positive on $[-1, 1]$. Theorem 2.3 can be applied to $\{L_n\}$.

3. APPLICATIONS

3.1. Variation Diminishing Splines

Let $n \geq 1, m \geq 2$ be integers and $l = m + n - 2$. Let $N_j(x), j = 0, 1, \dots, l$, be the functions defined by Marsden and Schoenberg [13, p. 66], and let ξ_j be the nodes defined by [13, p. 66]. With a function $f(t)$ defined on $[0, 1]$, associate the B -splines

$$S_{m,n}(f; x) = \sum_{j=0}^l f(\xi_j) N_j(x), \tag{3.1}$$

where $0 \leq x \leq 1$, having knots $x_0 = 0, x_1 = 1/n, \dots, x_n = 1$, and degree $m - 1$ (see [13, p. 68]). As noted in [13], the operators (3.1) are positive on $[0, 1]$.

Let $\mu_l^2(x) = S_{m,n}((t-x)^2; x)$. We shall need the following result, which combines Lemma 3, relation (4.13) and relation (2.4) of [13].

LEMMA 3.1. (i) $S_{m,n}(e_0; x) = 1, 0 \leq x \leq 1;$

(ii) if $3 \leq m \leq n + 2$, then $\mu_l^2 \leq \frac{m}{12n^2}, 0 \leq x \leq 1;$

(iii) if $m > n + 2$, then $\mu_l^2 \leq \frac{1}{4(m-2)}, 0 \leq x \leq 1;$ and

(iv) if $m = 2$, then $\mu_l^2 \leq \frac{1}{4n^2}, 0 \leq x \leq 1.$

THEOREM 3.2. Let $1/p + 1/p' = 1$, where $p > 1$. Let $0 < \alpha \leq x \leq \beta < \infty$ and $f \in C[0, \infty]$. Let k be a positive integer such that $f(x) = O(\exp_k(x) x^{2 \cdot p'} + x^2), x \rightarrow \infty$. Choose J such that $x \in [0, \ln_{k+1} J]$. Let $\delta > 0$ be such that $0 < \alpha - \delta$. Let $\alpha_l = \ln_{k+1}(l)$. Then:

(i) if $3 \leq m \leq n + 2$ and $l \geq J$,

$$\begin{aligned} & \left| S_{m,n} f(\alpha_l t; \frac{x}{\alpha_l}) - f(x) \right| \\ & \leq 2\omega_\delta \left(f, \frac{\ln(l)}{n} \left(\frac{m}{12} \right)^{1/2} \right) \\ & \quad + \left(\frac{\ln(l)}{n} \left(\frac{m}{12} \right)^{1/2} \right)^2 \left[\frac{\|f\|}{\delta^2} + \frac{C_1(f) M(\delta, \alpha, \beta)}{\delta^2} \right] \\ & \quad + \frac{C_2(f) M^{1/p'}}{\delta^{2/p'}} \left(\frac{\ln(l)}{n} \left(\frac{m}{12} \right)^{1/2} \right)^{2 \cdot p'} \cdot O(1), \end{aligned} \tag{3.2}$$

as $l \rightarrow \infty$, where $\|f\|$ and the constants have the same meaning as in Theorem 2.3 and $\omega_\delta(f, \cdot)$ is the modulus of continuity of f over $[\alpha - \delta, \beta + \delta] \subset [0, \infty)$.

(ii) if $m > n + 2$ and $l \geq J$, then $(\ln(l)/n) (m/12)^{1/2}$ in (3.2) is replaced by $\ln(l)/(2(m-2)^{1/2});$

(iii) if $m = 2$ and $l \geq J$, then $(\ln(l)/n) (m/12)^{1/2}$ in (3.2) is replaced by $\ln(l)/2n.$

Proof. Use Lemma 3.1, Theorem 2.3 and Remark 2.

Theorem 3.2 improves Theorem 4.5 of [4], which was stated for $f(x) = O(\exp(x))$. For the choice $n = 1$, we obtain an estimate for the Bernstein polynomial of degree $m - 1$ under multiplier enlargement. In this particular case, Theorem 3.2 improves Chlodovsky's result [11, p. 36], which was established for $f(x) = O(\exp(x))$.

3.2. Convolution Operators

Let $\omega(x)$ be non-negative, even and Lebesgue integrable on $[-1, 1]$. Assume ω is bounded and bounded away from 0, on closed subintervals of $(-1, 1)$. Let $\{Q_n(x)\}$ be the sequence of orthonormal polynomials on $[-1, 1]$ corresponding to $\omega(x)$. Let $x_{1,2n}$ and $x_{2,2n}$ be the two smallest positive zeros of $Q_{2n}(x)$. Define the sequence of polynomials, $\{\lambda_n(t)\}$, by

$$\lambda_n(t) = C_n \left(\frac{Q_{2n}(t)}{(t^2 - x_{1,2n}^2)(t^2 - x_{2,2n}^2)} \right)^2, \quad n = 1, 2, \dots,$$

where C_n is chosen so that $\int_{-1}^1 \lambda_n(t) dt = 1$. Define the sequence of operators $\{A_n\}$, by

$A_n(f; x) = \int_{-1/2}^{1/2} f(t) \lambda_n(t-x) dt$. Then $\{A_n\}$ is a sequence of positive linear operators from $C[-1/2, 1/2]$ to $C[-\eta, \eta]$, where $0 < \eta < \frac{1}{2}$.

In [2, p. 183] (see also [1]), it is shown that there are absolute constants $\alpha_0, \alpha_1, \alpha_2$ such that

$$\|e_i - A_n(e_i)\|_{[-\eta, \eta]} \leq \alpha_i/n^4, \quad i = 0, 1 \tag{3.3}$$

and

$$\mu_n^2 = \|A_n((t-x)^2; x)\|_{[-\eta, \eta]} \leq \alpha_2/n^2. \tag{3.4}$$

Furthermore, $\|A_n(e_0)\|_{[-\eta, \eta]} < 1$ for $n = 1, 2, \dots$. The operators A_n provide an optimal order of approximation to function in $C[-\eta, \eta]$ by means of linear, positive, algebraic polynomial operators [2, Chap. 6].

Combining (3.3), (3.4) and Theorem 2.3, we have

THEOREM 3.3. *Let $f \in C(-\infty, \infty)$ and let k be a positive integer and $p' > 1$ such that $f(x) = O(\exp_k |k||x|^{2/p'} + x^2)$, $|x| \rightarrow \infty$. Let $\alpha_n = \ln_{k+1}(n)$. Let $\alpha \leq x \leq \beta$ and let δ, η be positive numbers with $0 < \eta < \frac{1}{2}$. Choose N such that $x \in [-\eta\alpha_N, \eta\alpha_N]$. Then, if $\{A_n\}$ is the sequence of operators defined above and if $n \geq N$, we have*

$$\begin{aligned} & \left| A_n \left(f(\alpha_n t); \frac{x}{\alpha_n} \right) - f(x) \right| \\ & \leq 2\omega_\delta \left(f; \frac{\ln(n)}{n} \sqrt{\alpha_2} \right) \\ & \quad + \left(\frac{\ln(n)}{n} \sqrt{\alpha_2} \right)^2 \left(\frac{\|f\|}{\delta^2} + \frac{C_1(f) M(\delta, \alpha, \beta)}{\delta^2} \right) \\ & \quad + \|f\| \frac{\alpha_0}{n^4} + \frac{C_2(f) M^{1/p'}}{\delta^{2/p'}} \left(\frac{\ln(n)}{n} \sqrt{\alpha_2} \right)^{2/p'} \cdot O(1), \end{aligned}$$

as $n \rightarrow \infty$, where α_0, α_2 are from (3.3) and (3.4) and $\omega_\delta, \|f\|$ and the remaining constants are as in Theorem 2.3.

In Theorem 3.3, if $1 < p' \leq 2$, then the rate of convergence is governed by $1/n^{1-\epsilon}$ for any $\epsilon, 0 < \epsilon < 1$. In the case of $|A_n(f(t); x) - f(x)|$, where $f \in C[-1/2, 1/2]$ and $x \in [-\eta, \eta]$, the rate of convergence is governed by $1/n$. In view of the fact that $\{A_n(f(t); x)\}$ is optimal on $[-\eta, \eta]$, the estimate given by Theorem 3.3 is possibly the best that can be obtained for positive, polynomial operators under multiplier enlargement.

3.3. An Optimal Discrete Polynomial Operator

Let $\{Q_n(t)\}$ denote the sequence of orthonormal Legendre polynomials on $[-1, 1]$ with weight function $\omega(x) \equiv 1$. If $P_n(t) = Q_n(2t - 1)$, then $\{P_n(t)\}$ is orthonormal on $[0, 1]$ with weight function $\omega(x) \equiv 2$. Let $0 < x_{1,n} < x_{2,n} < \dots < x_{n,n} < 1$ be the zeros of $P_n(t)$. Let $y_{1,2n}$ and $y_{2,2n}$ be the two smallest positive zeros of $Q_{2n}(t)$. Let $R_n(t)$ be defined by

$$R_n(t) = C_n \left(\frac{Q_{2n}(t)}{(t^2 - y_{1,2n}^2)(t^2 - y_{2,2n}^2)} \right)^2, \quad n = 1, 2, \dots,$$

where C_n is chosen so that

$$\int_{-1}^1 R_n(t) dt = 1. \tag{3.5}$$

For each $n = 1, 2, \dots$, let $\lambda_{k,2n}, k = 1, 2, \dots, 2n$, be the Cotes numbers which arise in the Gauss quadrature formula applied to $[0, 1]$ and based on the zeros of $P_{2n}(t)$. Let $\mu_{k,2n}, k = -n, -n + 1, \dots, -1, 1, 2, \dots, n$, be the Cotes numbers which arise in the Gauss quadrature formula applied to $[-1, 1]$ and based on the zeros of $Q_{2n}(t)$ (see, e.g., [18, p. 47]).

For each $n = 1, 2, \dots$ define the operator K_n by

$$K_n(f(t); x) = \frac{1}{2} \sum_{k=1}^{2n} f(x_{k,2n}) \lambda_{k,2n} R_n(x_{k,2n} - x). \tag{3.6}$$

K_n is a positive linear operator from $C[0, 1]$ to $C[\eta, 1 - \eta], 0 < \eta < \frac{1}{2}$. In [17], it is shown that the sequence, $\{K_n\}$, is optimal in the sense of DeVore [2, Chap. 6]. Specifically, there are absolute constants, α_0, α_1 and α_2 , such that

$$\|K_n(e_i) - e_i\|_{[\eta, 1-\eta]} \leq \alpha_i/n^4, \quad i = 0, 1, n = 1, 2, \dots \tag{3.7}$$

and

$$\mu_n^2 = \|K_n((t-x)^2; x)\|_{[\eta, 1-\eta]} \leq \alpha_2/n^2, \quad n = 1, 2, \dots \tag{3.8}$$

In addition, $\|K_n(e_0)\| < 1$ for $n = 1, 2, \dots$. We also note that the constant C_n defined by (3.5) is given by

$$\frac{1}{C_n} = \sum_{\substack{j=-2 \\ j \neq 0}}^2 \mu_{j,2n} \beta_j,$$

where β_j denotes the value of $(Q_{2n}(t)/(t^2 - y_{1,2n}^2)(t^2 - y_{2,2n}^2))^2$ at $y_{j,2n}$.

THEOREM 3.4. *Let δ, η be positive numbers with $0 < \eta < \frac{1}{2}$. Let $f \in C(-\infty, \infty)$ and choose a positive integer k and $p' > 1$ such that $f(x) = O(\exp_k |x| |x|^{2/p'} + x)$. Let $\alpha_n = \ln_{k+1}(n)$. Let $-\infty < \alpha \leq x \leq \beta < \infty$. Choose N so that $x \in [-\alpha_N, \alpha_N]$. For the operators, K_n , defined by (3.6) we have*

$$\begin{aligned} & \left| K_n \left(f \left(\alpha_n g^{-1}(t), g \left(\frac{x}{\alpha_n} \right) \right) - f(x) \right| \\ & \leq 2\omega_\delta \left(f, \frac{2 \ln(n) \sqrt{\alpha_2}}{(1 - 2\eta) n} \right) \\ & \quad + \left(\frac{2 \ln(n) \sqrt{\alpha_2}}{(1 - 2\eta) n} \right)^2 \left[\frac{\|f\|}{\delta^2} + \frac{C_1(f) M(\delta, \alpha, \beta)}{\delta^2} \right] \\ & \quad + \|f\| \frac{\alpha_0}{n^4} + \frac{C_2(f) M^{1/p'}}{\delta^{2/p'}} \left(\frac{2 \ln(n) \sqrt{\alpha_2}}{n(1 - 2\eta)} \right)^{2/p'} \cdot O(1), \end{aligned}$$

as $n \rightarrow \infty$, where α_0, α_2 are from (3.7) and (3.8), g is the linear transformation mapping $[-1, 1]$ to $[\eta, 1 - \eta]$ and $\omega_\delta, \|f\|$ and the remaining constants are as in Theorem 2.3.

Proof. Use (3.7), (3.8) and Theorem 2.3, and Remark 5.

The comments following Theorem 3.3 apply equally well to Theorem 3.4.

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